Dielectric-breakdown-type crack-growth models and the fractal distribution of earthquake faults

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The two-dimensional dielectric-breakdown-type crack-growth models are studied and used to model the formation of earthquake faults. It is found that three kinds of earthquake faults exist, including modes II and III as well as a mixture of modes II and III, with $D=D_{\rm II}=2$, $D=D_{\rm III}=1.71$, $D=P_{\rm II}D_{\rm II}+P_{\rm III}D_{\rm III}=1.71$–2, respectively. These predictions are in good agreement with the experimental data.

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I. INTRODUCTION

Since the introduction of the model of diffusion-limited aggregation (DLA) [1], the extensive studies have proved that this model correctly describes many physical phenomena, such as dielectric breakdown (DB) [2], viscous fingering, electrodeposition, and dendritic growth [3]. Recently, crack propagation has also been considered from this point of view [4]. But DB-type growth patterns do not appear in many crack propagation phenomena. The reason is probably that due to the Griffith's condition, the crack will evolve from DB-type patterns (at small sizes) into spiky structures (at large sizes) [5]. But in the earth's crust the stresses are very high and the Griffith's condition is not important. In this case we expect that DB-type growth patterns should appear. It has been found [6,7] that the spatial distribution of earthquakes is fractal. In particular, in a laboratory model of the earth's crust Sornette, Davy, and Sornette [8] found that complex fractal patterns of faults are formed with $D=1.70\pm0.05$, suggesting a striking resemblance with DB. In this paper DB-type crack-growth (DBCG) models are introduced to model the formation of earthquake faults.

II. DB-TYPE CRACK-GROWTH MODELS

Most earthquakes occur in the brittle surface layer of the earth's crust with a depth of 10–20 km. This brittle layer can be considered to be two dimensional (2D). We introduce the 2D DBCG models [4] to model the formation of earthquake faults. For simplicity, the medium is assumed to be elastic and isotropic. Crack-growth speeds V_n are related to the tangential tensions τ at the crack surfaces by $V_n \sim |\tau|^\eta$ $(0\le\eta<\infty)$. Since for DB the case $\eta=1$ is the most realistic case for the DB experiment [2], for DBCG we naturally let $\eta=1$.

In order to make predictions of fractal dimensions for DBCG, we first review the Turkevich-Scher theory [9]. This theory determines the fractal dimension of a DLA cluster from the growth probability of the maximally extended portion of the cluster. Consider a DLA cluster, if one growth takes place at the tip of the maximally extended portion, then the radius of the cluster becomes $R\to R+a$, here a being the inner cutoff length. The probability that this event takes place is $P_{\max}\sim R^{-\alpha_{\min}}$, $\alpha_{\min}$ being the growth exponent. On the other hand, if this growth takes place elsewhere, the radius of the cluster remains unchanged. Thus if dN growths take place, on the average, $dN\,P_{\max}$ growths will take place at the tip of the maximally extended portion, then

$$dR=a\,dN\,P_{\max}\ . \tag{1}$$

Making use of $N\sim R^D$, we obtain fractal dimension

$$D=1+\alpha_{\min}\ . \tag{2}$$

This relation is well founded and confirmed by computer simulations.

Furthermore, $\alpha_{\min}$ can be related to the singularity exponent of the electric field at the tip of the maximally extended portion of the cluster. The DB problems are governed by Laplace equation $\Delta\Phi=0$, Φ being the electric potential, at the cluster $\Phi=0$, at infinity $\Phi=1$. At the surface of the cluster, the growth speeds $V_n\sim|\vec\nabla\Phi\cdot\vec n|$. For a coarse-grained description the maximally extended portion of the cluster can be modeled as a wedge with the angle θ, as shown in Fig. 1. The electric field at the wedge tip can be solved exactly by conformal mapping method. The result is

$$|\vec\nabla\Phi\cdot\vec n|\sim r^{\delta}\ , \tag{3}$$

where $\delta=-(\pi-\theta)/(2\pi-\theta)$.

Thus the growth probability at the wedge tip is

$$P_{\max}\sim\frac{\int_0^a|\vec\nabla\Phi\cdot\vec n|dr}{\int_0^\xi|\vec\nabla\Phi\cdot\vec n|dr}\sim(a/\xi)^{1+\delta}\ , \tag{4}$$

where ξ is an outer cutoff length. For a self-similar cluster $\xi\sim R$, thus

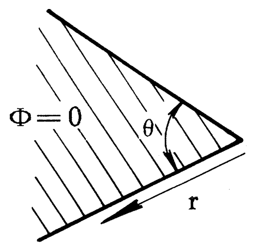

FIG. 1. The wedge.

$$P_{max} \sim R^{-(1+\delta)} , \tag{5}$$

$$D = 2+\delta = \frac{3\pi-\theta}{2\pi-\theta} . \tag{6}$$

The wedge angle θ cannot be determined from a first principle. By comparison with simulation value $D=1.71$, we find $\theta=42\pi/71$.

The above procedure can be extended to the 2D DBCG problems. The crack problems can be decomposed into mode I (opening mode), II (sliding mode), as well as III (tearing mode) [10]. For a coarse-grained description the maximally extended portion of a 2D DBCG cluster can be modeled as a sharp notch with the angle φ. The stresses near the notch tip are

$$\sigma_{ij} \sim r^{\lambda} \tag{7}$$

where λ is the stress exponent (see the Appendix). λ is given by, for modes II and I,

$$\sin(\lambda+1)(2\pi-\varphi) = \pm(\lambda+1)\sin(2\pi-\varphi) \tag{8}$$

and for mode III

$$\lambda = \frac{n\pi+\varphi}{2\pi-\varphi} \quad (n=-1,0,1,2,\ldots) . \tag{9}$$

The dominant contribution to the notch tip stress and displacement fields occurs for $\lambda_{min} > -1$. The curves of λ_{min} as a function of φ are shown in Fig. 2. Since for DB,

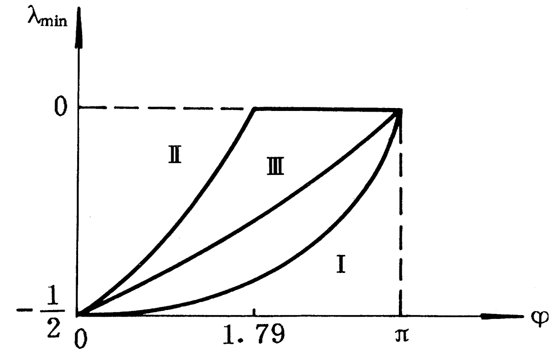

FIG. 2. λ_{min} as a function of φ for modes I, II, and III.

$V_n \sim |\vec{\nabla}\Phi\cdot\vec{n}|$ and for DBCG, $V_n \sim |\tau|$, then from Eqs. (3) and (7), we find $\delta \rightleftharpoons \lambda_{min}$. Thus from Eq. (6), we obtain the fractal dimensions of DBCG of three modes

$$D_{mode} = 2+\lambda_{min}(\varphi) . \tag{10}$$

If the boundary conditions at infinity are uniform, the notch angle φ of DBCG of three modes should be equal to the wedge angle θ of DB, i.e., $\varphi=\theta=42\pi/71$. This assumption comes from this argument: Since both DB and DBCG models of mode III are governed by the same growth law and the Laplace equation, if at infinity both models have the same uniform boundary conditions, they should belong to the same universality class. The validity of this assumption can be confirmed later. In this case the fractal dimensions of DBCG of three modes are $D_{mode}=2+\lambda_{min}$ ($\varphi=42\pi/71$), i.e., $D_{I}=1.575, D_{II}=2$, $D_{III}=1.71$. The fractal dimensions of DBCG are $D=P_{I}D_{I}+P_{II}D_{II}+P_{III}D_{III}$, here P_{I}, P_{II}, and P_{III} being the probabilities that modes I, II, and III appear, respectively.

If the boundary conditions at infinity are not uniform, the crack will grow faster in certain preferred directions.

TABLE I. Summary of DB and DBCG models.

		DBCG	
	DB	III	I,II
Equations	$\Delta\Phi=0$	$\Delta u_3=0$	$\Delta\Delta\chi=0$
Growth laws	$V_n \sim \vert\vec{\nabla}\Phi\cdot\vec{n}\vert$	$V_n \sim \vert\tau\vert$	
Fields at the wedge or notch dip	$\vert\vec{\nabla}\Phi\cdot\vec{n}\vert \sim r^{\delta}$ $\delta=-\frac{\pi-\theta}{2\pi-\theta}$	$\sigma_{ij} \sim r^{\lambda}$ $\lambda=\frac{\varphi+n\pi}{2\pi-\varphi}$ $(n=-1,0,1,2,\ldots)$	$\sin(\lambda+1)(2\pi-\varphi)$ $=\pm(\lambda+1)\sin(2\pi-\varphi)$ (for modes II and I)
Fractal dimensions	$D=2+\delta$	$D_{mode}=2+\lambda_{min}(\varphi)$ $D=P_{I}D_{I}+P_{II}D_{II}+P_{III}D_{III}$	

The cluster will evolve into a needle shape. The asymptotic notch angle $\varphi \to 0$, $D_{\text{mode}} = 2 + \lambda_{\min}(\varphi) \to \frac{3}{2}$, $D \to \frac{3}{2}$. Both models are summarized in Table I.

Let us compare these predictions with computer simulations. Louis and co-workers [4] have performed extensive computer simulations of 2D DBCG models on a triangular lattice. Under a uniform dilation boundary condition it was found that $D = 1.55$. In the continuum limit the crack growth belongs to crack mode I. The theoretical value of fractal dimension is $D = 1.575$. Under a shear boundary condition it was found that $D = 1.60$. Taguch [11] performed simulations of 2D DBCG models of mode III. A shear boundary condition was used. It was found that $D = 1.55 \pm 0.05$. In the latter two cases the boundary conditions are not uniform. The theoretical values of the asymptotic fractal dimensions are $D = \frac{3}{2}$. From these examples we conclude that the theoretical values of fractal dimensions of 2D DBCG models are in good agreement with simulations.

III. THE FRACTAL DISTRIBUTION OF EARTHQUAKE FAULTS

Kagan and Knopoff [6] studied the spatial distribution of earthquakes and found $D \approx 2$, which they interpreted as meaning that earthquakes occur on planes. Also, Sahimi, Robertson, and Sammis [7] analyzed the seismic data and found that the spatial distribution of hypocenters is fractal over at least one order of magnitude with $D \approx 1.8$, for which they proposed a percolation interpretation.

Now we use the 2D DBCG models to model the formation of earthquake faults that locate in the brittle surface layer of the earth's crust. In the crust the stresses are compressible. Thus earthquake faults of crack mode I do not exist. The boundary conditions at infinity can be regarded as the uniform. In general, the stress fields of earthquake faults are those of mixed-mode problems of modes II and III. Therefore, there exist three kinds of earthquake faults: (1) mode II, with $D = D_{\text{II}} = 2$, which we believe to be what Kagan and Knopoff [6] discovered; (2) mode III, with $D = D_{\text{III}} = 1.71$; (3) mixture of modes II and III, with $D = P_{\text{II}} D_{\text{II}} + P_{\text{III}} D_{\text{III}} = D_{\text{III}} \sim D_{\text{II}} = 1.71-2$, which we believe to be what Sahimi *et al.* [7] discovered. Here P_{II} and P_{III} are the probabilities that modes II and III appear, respectively. P_{II} and P_{III} depend on the stress fields of the earth's crust. Most earthquake faults belong to the mixed-mode faults. The theoretical values of fractal dimensions of earthquake faults are in good agreement with the experimental results.

IV. CONCLUSION

The Turkevich-Scher theory is extended to study the 2D DBCG models. It is found that if the boundary condition at infinity is not uniform, the cluster will evolve into a needle shape with $D = \frac{3}{2}$; if uniform, the models can be decomposed into modes I, II, and III, with $D_{\text{I}} = 1.575$, $D_{\text{II}} = 2$, $D_{\text{III}} = 1.71$, $D = P_{\text{I}} D_{\text{I}} + P_{\text{II}} D_{\text{II}} + P_{\text{III}} D_{\text{III}}$. These predictions are consistent with computer simulations. The models are used to model the formation of earthquake faults. It is predicted that there exist three kinds of earthquake faults, including models II and III as well as a mixture of modes II and III, with $D = D_{\text{II}} = 2$, $D = D_{\text{III}} = 1.71$, $D = P_{\text{II}} D_{\text{II}} + P_{\text{III}} D_{\text{III}} = 1.71-2$, respectively. These predictions are in good agreement with the experimental data.

APPENDIX

We calculate exponents of stress fields at a sharp notch tip. The notch coordinate systems are shown in Fig. 3.

For mode III, $u_1 = 0$, $u_2 = 0$, $u_3 = u_3(x_1, x_2)$, and the Lamé equation reduces to the Laplace equation [10,12]

$$\Delta u_3 = 0 \ . \tag{A1}$$

According to the complex variable theory, for any analytic function $f(z) = u(x_1, x_2) + iv(x_1, x_2)$, its real and imaginary parts satisfy the Laplace equation. Thus we let

$$u_3 = \frac{1}{\mu}[f(z) + \bar{f}(\bar{z})] \ , \tag{A2}$$

where μ is the Poisson's ratio, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, and $\bar{f}(\bar{z}) = u(x_1, x_2) - iv(x_1, x_2)$. If we use variables $z, \bar{z}$ to replace x_1, x_2, thus strain components become

$$\begin{aligned} \varepsilon_{31} &= \frac{1}{2\mu}[f'(z) + \bar{f}'(\bar{z})] \ , \\ \varepsilon_{32} &= \frac{i}{2\mu}[f'(z) - \bar{f}'(\bar{z})] \ . \end{aligned} \tag{A3}$$

Therefore

$$\sigma_{31} - i\sigma_{32} = 2f'(z) \ . \tag{A4}$$

Consider the analytic function

$$f(z) = (A + iB) z^{\lambda+1} \ , \tag{A5}$$

where A, B, and λ are real constants to be determined. For finite displacements at the notch tip, $\lambda > -1$. The substitution of Eq. (A5) into Eq. (A4) yields

$$\sigma_{31} - i\sigma_{32} = 2(\lambda+1) r^{\lambda} (A + iB)(\cos\lambda\theta + i \sin\lambda\theta) \ ,$$

thus

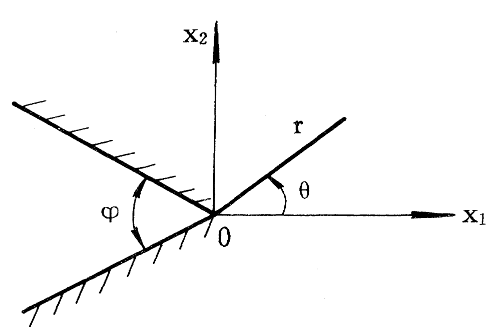

FIG. 3. The notch coordinate systems.

$$\sigma_{31}=2(\lambda+1)r^{\lambda}(A\cos\lambda\theta-B\sin\lambda\theta)\ ,$$
$$\sigma_{32}=-2(\lambda+1)r^{\lambda}(A\sin\lambda\theta+B\cos\lambda\theta)\ . \tag{A6}$$

The boundary condition that crack surfaces be traction free requires that

$$\left[\sigma_{32}\pm\sigma_{31}\tan\frac{\varphi}{2}\right]_{\theta=\pm(\pi-\varphi/2)}=0\ . \tag{A7}$$

This leads to

$$a_1A+a_2B=0,\quad a_1A-a_2B=0\ , \tag{A8}$$

where $a_1=\sin\lambda\theta_0-\tan\varphi/2\cos\lambda\theta_0$, $a_2=-\cos\lambda\theta_0-\tan\varphi/2\sin\lambda\theta_0$, $\theta_0=\pi-\varphi/2$. Thus we have

$$\begin{vmatrix} a_1 & a_2 \\ -a_1 & a_2 \end{vmatrix}=0 \tag{A9}$$

or $\sin(2\lambda\theta_0-\varphi)=0$. So

$$\lambda=\frac{\varphi+n\pi}{2\pi-\varphi}\quad (n=-1,0,1,2,\dots)\ . \tag{A10}$$

For a plane problem (modes I and II), $u_1=u_1(x_1,x_2)$, $u_2=u_2(x_1,x_2)$, and $u_3=0$, the equilibrium equations reduce to [10,12]

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}=0,\quad \frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}=0\ . \tag{A11}$$

The solutions to these equations are

$$\sigma_{11}=\frac{\partial^2\chi}{\partial x_2^2},\ \sigma_{12}=-\frac{\partial^2\chi}{\partial x_1\partial x_2},\quad \sigma_{22}=\frac{\partial^2\chi}{\partial x_1^2}\ , \tag{A12}$$

where $\chi=\chi(x_1,x_2)$ is called Airy stress function. Thus

$$\sigma_{11}+\sigma_{22}=\frac{E}{(1+\mu)(1-2\mu)}(\varepsilon_{11}+\varepsilon_{22})$$
$$=\frac{E}{(1+\mu)(1-2\mu)}\nabla\cdot\vec{u}$$
$$=\Delta\chi\ . \tag{A13}$$

Making use of the Lamé equation $\Delta\vec{\nabla}\cdot\vec{u}=0$, we have

$$\Delta\Delta\chi=0\ . \tag{A14}$$

Since $\Delta\chi$ satisfies the Laplace equation, thus

$$\Delta\chi=4\frac{\partial^2\chi}{\partial z\partial\bar{z}}=f(z)+\bar{f}(\bar{z})\ . \tag{A15}$$

Integrating this equation, we have

$$\chi=\frac{1}{2}[\bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z})]\ . \tag{A16}$$

where $\Omega(z)$ and $\omega(z)$ are analytic functions. Thus

$$\sigma_{11}+\sigma_{22}=2[\Omega'(z)+\overline{\Omega'}(\bar{z})]\ ,$$
$$\sigma_{22}-i\sigma_{12}=\Omega'(z)+\overline{\Omega'}(\bar{z})+z\overline{\Omega''}(\bar{z})+\overline{\omega''}(\bar{z})\ . \tag{A17}$$

The boundary condition is

$$\left[\sigma_{12}\pm\sigma_{11}\tan\frac{\varphi}{2}\right]_{\theta=\pm(\pi-\varphi/2)}=0\ ,$$
$$\left[\sigma_{22}\pm\sigma_{21}\tan\frac{\varphi}{2}\right]_{\theta=\pm(\pi-\varphi/2)}=0\ . \tag{A18}$$

For mode I, due to symmetry with respect to the notch plane, we choose a solution of the form

$$\Omega=Az^{\lambda+1},\quad \omega'=Bz^{\lambda+1} \tag{A19}$$

where A, B, and λ are real constants. For mode II, we choose

$$\Omega=iAz^{\lambda+1},\quad \omega'=iBz^{\lambda+1}\ . \tag{A20}$$

Following the same procedure, we obtain for modes II and I

$$\sin(\lambda+1)(2\pi-\varphi)=\pm(\lambda+1)\sin(2\pi-\varphi)\ . \tag{A21}$$

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